The spectrum of Yang Mills on a sphere

Alexander Barabanschikov,[a] Lars Grant,[bc] Lisa L. Huang[b] and Suvrat Raju[bc]

[a]*Department of Physics, Northeastern University*
Boston, MA 02115, U.S.A.

[b]*Jefferson Physical Laboratory, Harvard University*
Cambridge, MA 02138, U.S.A.

[c]*Tata Institute of Fundamental Research*
Mumbai 400005, India
E-mail: `abarabanschikov@yahoo.com`, `lgrant@fas.harvard.edu`,
`llhuang@fas.harvard.edu`, `suvrat@physics.harvard.edu`

ABSTRACT: In this note, we determine the representation content of the free, large N, SU(N) Yang Mills theory on a sphere by decomposing its thermal partition function into characters of the irreducible representations of the conformal group SO(4, 2). We also discuss the generalization of this procedure to finding the representation content of $\mathcal{N} = 4$ Super Yang Mills.

Contents

1. Introduction

Almost thirty years ago t'Hooft, Polyakov, Migdal and Wilson suggested that large N Yang Mills theory could be recast as a string theory. Electric flux tubes of the confining gauge theory were expected to map to dual fundamental strings. This picture seemed crucially tied to confinement and strongly coupled gauge dynamics; in particular Yang Mills perturbation theory was not expected to capture the stringy behavior of Yang Mills theory.

Some of these expectations were modified after one gauge-string duality was understood in detail via the celebrated AdS/CFT conjecture [1, 2]. According to the Maldacena conjecture, maximally supersymmetric SU(N) Yang Mills theory on an S^3 is dual to type IIB theory on $AdS_5 \times S^5$. $\mathcal{N} = 4$ Yang Mills theory is a conformal (rather than confining) gauge theory, further it possesses a tunable coupling constant λ which may smoothly be taken to zero. Nonetheless, the AdS/CFT conjecture asserts that this gauge theory is dual to a string theory at all values of λ including $\lambda = 0$.

It follows that stringy aspects of the $\mathcal{N} = 4$ theory must be visible even at arbitrarily weak coupling. Evidence has been accumulating over recent years that this is indeed the case. First, the spectrum of weakly coupled $\mathcal{N} = 4$ Yang Mills theory on an S^3 displays a Hagedorn like growth in its density of states [3, 4] (a distinctly stringy feature).

Second, perturbative contributions to the anomalous dimensions of certain long single trace operators are effectively encoded in a 1+1 dimensional field theory [5]. For recent developments in this story, please see [6] and references therein.

With these lessons in mind, it is natural to wonder if the stringy spectrum of the simplest of all large N Yang Mills theories — pure SU(N) Yang Mills — may also be computed (in some regime of parameters) in perturbation theory. Naively it seems that asymptotically free pure Yang Mills lacks a dimensionless parameter in which one might attempt a perturbative expansion. However Yang Mills theory on an S^3 of radius R does have a dimensionless coupling $\lambda = \Lambda R$, where Λ is the dynamically generated scale. When λ is taken to zero, the spectrum of pure Yang Mills may reliably be studied in perturbation theory, and turns out to display the same Hagedorn growth with energy as its supersymmetric counterpart [3, 4]. It thus seems plausible that, like its supersymmetric counterpart, pure Yang Mills theory on an S^3 has a dual string description at all values of λ.

In this note we will compute the spectrum of particles of the as yet unknown string dual to pure Yang Mills on a sphere at $\lambda = 0$. We do this by decomposing the partition function of our theory at $\lambda = 0$ (computed in [3, 4]) into a sum over characters of the conformal group (the conformal group is a good symmetry of pure Yang Mills theory precisely at $\lambda = 0$). According to the AdS/CFT dictionary, at large N, representations of the conformal group are in one to one correspondence with particles of the dual string theory; consequently our decomposition determines particle spectrum of interest.

The plan of the rest of this note is as follows. In sections 1–4 we review the conformal algebra and calculate the characters of the representations. Since this involves some subtleties, in section 5, we verify the calculation using the oscillator construction proposed in [7]. The integral we need to perform to obtain the character decomposition has poles. Section 6 discusses the appropriate treatment for these, while the actual calculation is performed in section 7. We conclude with some comments on the supersymmetric case. In the appendix we prove that the multiplicities obtained in section 7 are integers.

While we were in the process of writing this paper, we received [8] that has overlaps with sections 3, 4.

2. Conformal algebra

Adding dilatations and special conformal transformations to a set of Lorentz generators in 4 dimensions gives the conformal algebra.

$$
\begin{aligned}
[D, P_\mu] &= -iP_\mu, \\
[D, K_\mu] &= iK_\mu, \\
[K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D + M_{\mu\nu}), \\
[M_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \\
[M_{\mu\nu}, K_\rho] &= i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu), \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i\left(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\sigma}M_{\mu\rho}\right).
\end{aligned} \tag{2.1}
$$

We are interested in unitary representations of this algebra, where these generators are hermitian. However, it is convenient for the purposes of constructing the representations of this algebra, to choose a basis of generators which satisfies the euclidean conformal algebra [9] and in which the generators are no longer all hermitian. The generators (some D', P'^μ, etc.) in this new basis will satisfy the same algebra as above with $\eta_{\mu\nu} \to \delta_{\mu\nu}$. The hermiticity properties of the generators in this basis are:

$$M'^\dagger = M',$$
$$D'^\dagger = -D',$$
$$P'^\dagger = K',$$
$$K'^\dagger = P'.$$

From now on, we will use this new set of generators and drop the primes for clarity.

We can extract two sets of SU(2) generators from the Lorentz generators $M_{\mu\nu}$. We define:

$$J_1^z = 1/2(M_{12} + M_{03}),$$
$$J_2^z = 1/2(M_{12} - M_{03}),$$
$$J_1^+ = 1/2(M_{23} + M_{01} + i(M_{02} - M_{31})),$$
$$J_2^+ = 1/2(M_{23} - M_{01} - i(M_{31} + M_{02})),$$
$$J_1^- = J_1^{+\dagger}, \quad J_2^- = J_2^{+\dagger}. \tag{2.2}$$

We will also choose to use an hermitian operator $D'' = iD$ for convenience. We note that the set of generators $M = \{D'', J_1^z, J_1^+, J_1^-, J_2^z, J_2^+, J_2^-\}$ generate the maximal compact subgroup $SO(2) \times SO(4) \subset SO(4,2)$ of the conformal group. We can divide the generators into three sets:[1] $G_0 = \{D, J_1^z, J_2^z\}, G_+ = \{J_1^+, J_2^+, P_\mu\}, G_- = \{K_\mu, J_1^-, J_2^-\}$. With this division, the Lie algebra above has the property that:

$$[g_0, g_+] = g'_+,$$
$$[g_0, g'_0] = g''_0,$$
$$[g_+, g_-] = g_0,$$
$$[g_0, g_-] = g'_-.$$

where anything with a subscript 0 belongs to linear combinations of operators in G_0 and similarly symbols with subscripts $+(-)$ belong to linear combinations of operators in $G_+(G_-)$. These relations make it clear that G_+ and G_- act like raising and lowering operators on the charges G_0. The operators in G_0 commute and we will use these as Cartan generators for the algebra.

It will be convenient to choose linear combinations of the operators in G_+ and G_- that diagonalize G_0. These combinations are:

$$P_w = P_1 + iP_2,$$

---

[1]In an alternate grading, we could include all the generators of the compact SO(4) in a set G'_0. The elements of G_0 wold have zero energy while the elements of $G'_+(G'_-)$ would have positive(negative) energy. Since the primary state is a *lowest energy* state, it is annihilated by the elements of G_-

$$P_{\overline{w}} = P_1 - iP_2,$$
$$P_z = P_3 + iP_4, P_{\bar{z}} = P_3 - iP_4. \tag{2.3}$$

These generators all have well defined weights under the Cartan generators G_0.

$$
\begin{array}{cccc}
 & D & J_1 & J_2 \\
J_1^+ & 0 & 1 & 0 \\
J_2^+ & 0 & 0 & 1 \\
P_w & 1 & \frac{1}{2} & \frac{1}{2} \\
P_{\overline{w}} & 1 & -\frac{1}{2} & -\frac{1}{2} \\
P_z & 1 & -\frac{1}{2} & \frac{1}{2} \\
P_{\bar{z}} & 1 & \frac{1}{2} & -\frac{1}{2}
\end{array}
\tag{2.4}
$$

3. Representations of the conformal group

Any irreducible representation of the conformal group can be written as some direct sum of representations of the compact subgroup SO(4) × SO(2):

$$R_{\text{SO}(4,2)} = \sum_i \oplus R_{\text{comp}}^i. \tag{3.1}$$

We are ultimately interested in the occurrence of these representations in the partition function of the conformal Yang-Mills gauge theory quantized on $S^3 \times R$; the hamiltonian of the theory is the dilatation operator D. The spectrum of this theory is bounded below and therefore we will be interested in representations of the conformal algebra where the values of the charge D are bounded below. Then there must be some term, R_{comp}^λ in the above sum that has the lowest dimension. This term has a highest weight state $|\lambda\rangle$ with weights $\lambda = (d, j_1, j_2)$. The K^μ operators necessarily annihilate all the states in R_{rmcomp}^λ because the K^μ have negative weight under the operator D. If we consider the operation of the P^μ on this set of states, we generate a whole representation of the conformal algebra with states:

$$[\lambda]_l = R_{\text{SO}(4,2)}^\lambda = \sum_{n_w n_{\overline{w}} n_z n_{\bar{z}}} P_w^{n_w} P_{\overline{w}}^{n_{\overline{w}}} P_z^{n_z} P_{\bar{z}}^{n_{\bar{z}}} \times R_{\text{comp}}^\lambda. \tag{3.2}$$

We will denote this set of states by $[d, j_1, j_2]_l$. A careful analysis [10] shows that, barring the trivial case, this representation is unitary if one of the following conditions holds on the highest weight state $|\lambda\rangle$:

$$(i) \quad d \geq j_1 + j_2 + 2 \quad j_1 \neq 0 \quad j_2 \neq 0, (ii) \quad d \geq j_1 + j_2 + 1 \quad j_1 j_2 = 0. \tag{3.3}$$

In the case where equality does not hold in these unitarity conditions, the representation is called *long* and all the states produced by the operation of the P^μ are non-zero.

If equality holds in one of the conditions, then the representation will be a truncated *short* representation in which some of the states listed in (3.2) are 0. A unitary representation is one where we can define a positive definite norm. To find the states that should be absent in a short representation, one can assume that the states in R_{comp}^λ are normalized in

the standard way [9]. Calculating the norm of the states $P^\mu|\lambda\rangle$ will show that when equality holds in the unitarity conditions above, some of these states, say $a_\mu P^\mu|\lambda\rangle$ have norm 0. This should be interpreted as meaning that this state is 0 so that the operator $a_\mu P^\mu$ annihilates $|\lambda\rangle$. The descendants of $a_\mu P^\mu|\lambda\rangle$ then also do not occur in the representation. This last statement needs some care as we will see.

We will list here the possible types of short representations:

- In the generic short representation, $j_1 \neq 0, j_2 \neq 0, d = j_1 + j_2 + 2$ the states of norm 0 occur at level 1. The state $|d+1, j_1 - \frac{1}{2}, j_2 - \frac{1}{2}\rangle$ is *not* found in the representation and its descendants are also not to be found in the representation. The set of all descendants of the state $\lambda' = |d+1, j_1 - \frac{1}{2}, j_2 - \frac{1}{2}\rangle$ is the same as $[d+1, j_1 - \frac{1}{2}, j_2 - \frac{1}{2}]_l$, so that we may write the generic short, irreducible representation as:

$$[d, j_1, j_2] = [d, j_1, j_2]_l - \left[d+1, j_1 - \frac{1}{2}, j_2 - \frac{1}{2}\right]_l. \tag{3.4}$$

- In the case $j_1 = j_2 = 0$, $d = 1$, the state $|3, 0, 0\rangle$ is not found. All its descendants are also absent, so we may write the irreducible representation as

$$[1, 0, 0] = [1, 0, 0]_l - [3, 0, 0]_l. \tag{3.5}$$

This corresponds to the free massless scalar. The representation $[3, 0, 0]_l$ corresponds to the equation of motion $\partial_\mu \partial^\mu \phi = 0$.

- In the case $j_1 > 0, j_2 = 0$, the state $|d+1, j_1 - \frac{1}{2}, \frac{1}{2}\rangle$ is absent. Note that the weights of this state satisfy the unitarity bound (i) in (3.3). When we delete the states $[d+1, j_1 - \frac{1}{2}, \frac{1}{2}]_l$, we must delete it as a *short* representation, ie. we must not delete the states that do not occur in the short rep $[d+1, j_1 - \frac{1}{2}, \frac{1}{2}]$. We will do a calculation below, using an oscillator representation, showing that this is the correct way to remove the zero norm states in this case. We will have

$$[d, j_1, 0] = [d, j_1, 0]_l - \left[d+1, j_1 - \frac{1}{2}, \frac{1}{2}\right]_l + [d+2, j_1 - 1, 0]_l. \tag{3.6}$$

4. Characters

The characters for these representations are now easy to compute. First we compute a character of the set of states

$$[d, j_1, j_2]_l = \sum_{\substack{n_w n_{\bar{w}} n_z n_{\bar{z}} \\ |m1| < j_1 \\ |m2| < j_2}} \oplus P_w^{n_w} P_{\bar{w}}^{n_{\bar{w}}} P_z^{n_z} P_{\bar{z}}^{n_{\bar{z}}} |d, m_1, m_2\rangle. \tag{4.1}$$

We will denote the character of this set of states by a subscript l on χ:

$$\chi_{[d, j_1, j_2]}^l = \mathrm{Tr}_{[d, j_1, j_2]_l} \exp[iD\theta + iJ_1^z \theta_1 + iJ_2^z \theta_2] = \frac{\chi_{j_1}^{\mathrm{SU}(2)} \chi_{j_2}^{\mathrm{SU}(2)} e^{id\theta}}{\prod_{j=1}^4 (1 - \exp[i\vec{\alpha}_j \cdot \vec{\theta}])}. \tag{4.2}$$

where $\vec{\theta} = (\theta, \theta_1, \theta_2)$, and α_j runs over the 4 generators $P_w, P_{\bar{w}}, P_z, P_{\bar{z}}$ and refers to their weights taken from the table (2.4) , ie $\alpha_1 = (1, 1/2, 1/2)$.

The characters of the possible representations are given by:

LONG
$$(i) \quad d > j_1 + j_2 + 2 \qquad\qquad \chi_{[d,j_1,j_2]} = \chi^l_{[d,j_1,j_2]}$$
$$(ii) \quad d > j_1 + j_2 + 1, \ j_1 j_2 = 0 \quad \chi_{[d,j_1,j_2]} = \chi^l_{[d,j_1,j_2]}$$
SHORT
$$(iii) \quad j_1 = j_2 = 0, \ d = 1 \qquad\qquad \chi_{[1,0,0]} = \chi^l_{[1,0,0]} - \chi^l_{[3,0,0]}$$
$$(iv) \quad j_1 > 0, \ j_2 = 0, \ d = j_1 + 1 \ \ \chi_{[d,j_1,0]} = \chi^l_{[d,j_1,0]} - \chi^l_{[d+1,j_1-1/2,1/2]} + \chi^l_{[d+2,j_1-1,0]}$$
$$(v) \quad j_1, j_2 > 0, \ d = j_1 + j_2 + 2 \ \ \chi_{[d,j_1,j_2]} = \chi^l_{[d,j_1,j_2]} - \chi^l_{[d+1,j_1-1/2,j_2-1/2]} \qquad (4.3)$$

We will note shortly that these characters are *not* orthogonal. Nevertheless, they can be used to decompose the spectrum of the Conformal Yang Mills theory we are interested in.

5. Oscillator construction

Here we will discuss an oscillator construction [7] for the $SO(4,2)$ algebra and use it to confirm the character of the short representations $j_2 = 0$ and $d = j_1 + 1$. The $SO(4,2)$ algebra may be represented by 8 bosonic oscillators a^I, b^J, a_I and b_J ($I, J = 1, 2$) having the following commutation relations:

$$[a_I, a^J] = \delta_I^J \quad [b_P, b^Q] = \delta_P^Q. \qquad (5.1)$$

The generators of the $SO(4,2)$ group are represented as:

$$J_1^i = 1/2(\sigma^i)_I^J[a^I a_J - 1/2\delta_J^I a^K a_K], \quad J_2^i = 1/2(\bar{\sigma}^i)_P^Q[b^P b_Q - 1/2\delta_Q^P b^R b_R]$$
$$D = 1/2(N_a + N_b + 2), \quad P^{IJ} = a^I b^J, \quad K_{IJ} = a_I b_J. \qquad (5.2)$$

We note that a state constructed out of oscillators acting on a vacuum satisfying $a_I|0\rangle = b_J|0\rangle = 0$ has weights $(1/2(N_a + N_b + 2), 1/2(n_{a_1} - n_{a_2}), 1/2(n_{b_1} - n_{b_2}))$ under D, J_1^z, J_2^z (n_{a_1} is the number of a^1 operators used to create the state and $N_a = n_{a^1} + n_{a^2}$). The unitarity constraints are built into this representation, so we may calculate with it without worrying about states that have norm zero. For example, we may compute the "blind" partition function of the short representation $|\lambda\rangle = (j_1 + 1, j_1, 0)$. We first choose a state with the right weights to act as the primary:

$$(a^2)^{2j_1}|0\rangle. \qquad (5.3)$$

Now we can easily generate from this state, a representation of the maximal compact subgroup $SO(4) \times SO(2)$:

$$a^{I_1} a^{I_2} a^{I_3} \ldots a^{I_{2j_1}}|0\rangle. \qquad (5.4)$$

There are $2j_1 + 1$ states here, all with dimension $D = j_1 + 1$ as we expect. Now we operate with all possible P^μ to generate all states in the representation of the full conformal group:

$$\Psi_{\{I\},n_1,n_2,m_1,m_2} = a_1^{n_1} a_2^{n_2} b_1^{m_1} b_2^{m_2} a^{I_1} a^{I_2} a^{I_3} \ldots a^{I_{2j_1}}|0\rangle \qquad (5.5)$$

we note that $n_1 + n_2 = m_1 + m_2$ here because $P^{IJ} = a^I b^J$. The character of this representation is:

$$
\begin{aligned}
Z_{[j_1+1,j_1,0]} &= \sum_{\{I\},n_1,n_2,m_1,m_2} \left\langle \Psi^\dagger_{\{I\},n_1,n_2,m_1,m_2} \Big| x^D \Psi_{\{I\},n_1,n_2,m_1,m_2} \right\rangle \\
&= \sum_{N=0}^{\infty} x^{N+j_1} N(N+2j_1) \\
&= \frac{x^{j+1}(2j_1 + 1 - 4j_1 x + (2j_1 - 1)x^2)}{(1-x)^4}.
\end{aligned}
\tag{5.6}
$$

This agrees with the result in (4.3). In the second line, we have used the fact that $n_1 + n_2 = m_1 + m_2$ and that the number of a's in (5.4) is $2j_1$. This calculation can easily be repeated with chemical potentials added for the angular momenta.

6. Character decomposition

Character decomposition integrals are evaluated over the Haar measure of the group in question, in this case $SO(4,2)$. We can reduce these integrals to integrals over the maximal torus of the maximal compact subgroup $SU(2) \times SU(2) \times SO(2)$ using the Weyl integration formula [11]:

$$
\int_G f(g) d\mu_G = \frac{1}{|W|} \int_T f(t) \prod_{\alpha \in R} (1 - \exp(\alpha(t))) d\mu_T.
\tag{6.1}
$$

where $f(g)$ is a function satisfying $f(hgh^{-1}) = f(g)$ so that it only depends on the conjugacy class of g, and $d\mu_G$ and $d\mu_T$ are the Haar measures on the group G and the maximal torus T. $\alpha \in R$ means the product is over the roots of $SO(4,2)$. Each root corresponds to a generator in table (2.4), for example the factor corresponding to K_w is $(1 - \exp(-i(\theta + \frac{\theta_1+\theta_2}{2})))$. The constant $|W|$ is the order of the Weyl group in the compact case. In this non-compact case, it will diverge. We nevertheless obtain a useful orthogonality relation below where this constant is not relevant. An integral of characters over the group G becomes:

$$
\begin{aligned}
\int_G \chi^*_{[d,j_1,j_2]} \chi_{[d',j'_1,j'_2]} d\mu_G = \frac{1}{|W|} \int_0^{2\pi} \int_0^{4\pi} \int_0^{4\pi} \chi^*_{[d,j_1,j_2]}(\theta,\theta_1,\theta_2) \chi_{[d',j'_1,j'_2]}(\theta,\theta_1,\theta_2) \\
\prod_{\alpha \in R} (1 - \exp(i\vec{\alpha}\cdot\vec{\theta})) \frac{d\theta}{2\pi} \frac{d\theta_1}{4\pi} \frac{d\theta_2}{4\pi}.
\end{aligned}
\tag{6.2}
$$

While the characters of the non-compact group $SO(4,2)$ are not orthogonal, the characters of the sets of states $[d,j_1,j_2]_l$ can easily be shown to explicitly satisfy the following orthogonality relation:

$$
\frac{1}{4} \int \chi^{l*}_{[d,j_1,j_2]} \chi^l_{[d',j'_1,j'_2]} \prod_{\alpha \in R} (1 - \exp(i\vec{\alpha}\cdot\vec{\theta})) d\mu_T = \delta_{d,d'} \delta_{j_1,j'_1} \delta_{j_2,j'_2}.
\tag{6.3}
$$

This orthogonality is enough for us to decompose the partition function of YM into representations of the conformal group.

In the case of non-compact groups, character decomposition integrals involve some subtleties. Written naively, these integrals have poles. To learn how to deal with these poles, consider the representation $[1, 0, 0] \times [1, 0, 0]$. The decomposition of this tensor product by characters will involve integrals like

$$\int (\chi^l_{[1,0,0]})^2 \chi^{l*}_{[d,j_1,j_2]} d\mu_G =$$
$$\int \frac{(\cos j_1 \theta_1 - \cos(j_1+1)\theta_1)(\cos j_2 \theta_2 - \cos(j_2+1)\theta_2) \exp[-id\theta] \exp[2i\theta]}{\prod_{\alpha \in P}(1 - e^{i\vec{\alpha}\cdot\vec{\theta}})} \frac{d\theta}{2\pi} \frac{d\theta_1}{4\pi} \frac{d\theta_2}{4\pi}, \quad (6.4)$$

where $\alpha \in P$ means product only over the 4 roots corresponding to momentum generators P_i as in (4.2). It is clear that this integral has singularities along the contour of integration. To resolve this, we deform the contour *inwards*. This is equivalent to *ignoring* the contribution from the boundaries.

To see why this is justified, expand

$$\prod \frac{1}{1 - xq_i} = \sum x^n q_i^n. \quad (6.5)$$

We have introduced new notation here. $x = e^{i\theta}$, $q_i = e^{i\frac{\pm\theta_1\pm\theta_2}{2}}$ for $i = 1, 2, 3, 4$. Now recalling that x measures the scaling dimension or the energy, we see that we should add a small imaginary part to θ which is equivalent to inserting an energy cutoff in the integral.

With this pole prescription, the decomposition yields:

$$\chi_{[1,0,0]\otimes[1,0,0]} = \sum_{d=2}^{\infty} \chi_{[d,\frac{d-2}{2},\frac{d-2}{2}]}. \quad (6.6)$$

These representations are generically short (barring $[2, 0, 0]$).

We can count the operators in our theory manually to check this result. Using two scalar field representations, the primary operators in the tensor product at the first few levels are:

$$\begin{aligned} \phi_1\phi_2 & \quad [2, 0, 0], \\ \phi_2\partial_\mu\phi_1 - \phi_1\partial_\mu\phi_2 & \quad \left[3, \frac{1}{2}, \frac{1}{2}\right], \\ \partial_\mu\phi_1\partial_\nu\phi_2 + \partial_\nu\phi_1\partial_\mu\phi_2 & \quad [4, 1, 1]. \end{aligned} \quad (6.7)$$

which agrees with the decomposition. We will use this same pole prescription in performing the decomposition of the YM theory.

7. The integral

The single trace partition function of Free Yang Mills on a sphere was calculated in [3, 4]. The result was written as

$$Z[\theta, \theta_1, \theta_2] = -\sum \frac{\phi(k)}{k} \ln(1 - z(k\theta, k\theta_1, k\theta_2)), \quad (7.1)$$

where the 'single particle' partition function, z is given by:

$$z = 1 + \frac{(x - x^3)\sum_i q_i + x^4 - 1}{\prod_i (1 - xq_i)}. \tag{7.2}$$

and the Euler totient function, $\phi(k)$, is the number of integers less than k that are relatively prime to k. We need to decompose this expression as a sum of characters of the conformal group. First,

$$1 - z = \frac{(1 - x^2)(x^2 - (\sum q_i)x + 1)}{\prod_i (1 - xq_i)}. \tag{7.3}$$

So, the logarithm in (7.1) will separate the factors here into terms which we will integrate one at a time. Also, we have explicitly

$$\chi^l_{d,j_1,j_2} = \frac{\sin[(j_1 + 1/2)\theta_1]}{\sin[\frac{\theta_1}{2}]} \frac{\sin[(j_2 + 1/2)\theta_2]}{\sin[\frac{\theta_2}{2}]} \exp[id\theta] \prod_i \frac{1}{(1 - xq_i)}. \tag{7.4}$$

The measure of integration is:

$$dM = 4\sin^2[\frac{\theta_1}{2}]\sin^2[\frac{\theta_2}{2}] \prod_i (x - 1/q_i)(x - q_i) \frac{d\theta}{2\pi} \frac{d\theta_1}{4\pi} \frac{d\theta_2}{4\pi}. \tag{7.5}$$

Note that the θ_1, θ_2 integrals go over $0, 4\pi$.

We will evaluate

$$\int dM Z[\theta, \theta_1, \theta_2] \bar{\chi}^*_{[d,j_1,j_2]}. \tag{7.6}$$

Half of the measure cancels with the denominator of the character. The remaining part of the measure may be written as

$$\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} \prod_i (x - q_i) =$$

$$= 4\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} \left\{ (x^4 + 1) - 4\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}(x^3 + x) + 2x^2(\cos\theta_1 + \cos\theta_2 + 1) \right\}. \tag{7.7}$$

We will do the integral over the 4 linear factors in the denominator of (7.3) first. The contribution from the partition function Z is

$$-\sum_{k,i} \frac{\phi(k)}{k} \log\frac{1}{1 - x^k q_i^k} = -\sum_{k,i,n} \phi(k)\frac{x^{kn}q_i^{kn}}{kn} = -\sum_{k,n} \phi(k)\frac{4\cos\frac{kn\theta_1}{2}\cos\frac{kn\theta_2}{2}x^{kn}}{kn}. \tag{7.8}$$

The integration over θ picks out coefficients of x^d in the product of (7.8) and the measure (7.7). The coefficient of x^d in (7.8) is

$$c(d) = -\sum_{k|d} \frac{\phi(k)}{d} 4\cos\frac{d\theta_1}{2}\cos\frac{d\theta_2}{2} = -4\cos\frac{d\theta_1}{2}\cos\frac{d\theta_2}{2}. \tag{7.9}$$

Hence, we need to deal with the integral

$$\int \left[c(d) + c(d - 4) - 4\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}(c(d - 1) + c(d - 3)) + 2(\cos\theta_1 + \cos\theta_2 + 1)c(d - 2) \right]$$

$$\times \left\{ (\cos j_1\theta_1 - \cos(j_1 + 1)\theta_1)(\cos j_2\theta_2 - \cos(j_2 + 1)\theta_2)\frac{d\theta_1}{4\pi}\frac{d\theta_2}{4\pi} \right\}. \tag{7.10}$$

– 9 –

With $\Delta_b^a = \delta_b^a + \delta_b^{-a}$ the contribution from the factors in the denominator of (7.3) is given by

$$I_1[d, j_1, j_2] = -\Delta_{j_2}^{j_1}\left(\Delta_{2j_1}^d + \Delta_{2j_1}^{d-4} + 2\Delta_{2j_1}^{d-2}\right) + \Delta_{2j_1}^{d-1\pm1}\Delta_{2j_2}^{d-1\pm1}$$
$$+\Delta_{2j_1}^{d-3\pm1}\Delta_{2j_2}^{d-3\pm1} - \Delta_{2j_1}^{d-2\pm2}\Delta_{2j_2}^{d-2\pm2}. \tag{7.11}$$

Next, we consider the $(1-x^2)$ factor in (7.3).

$$-\log(1-x^{2k}) = \sum \frac{x^{2kn}}{n}. \tag{7.12}$$

This time, for the coefficient of x^d, we have

$$\sum_{2k|d} \frac{2\phi(k)}{d} = \left\{ \begin{array}{ll} 1 & \text{d even} \\ 0 & \text{otherwise} \end{array} \right. \tag{7.13}$$

Substituting this into the main integral, we find that for $d > 4$, we need to integrate

$$\begin{array}{ll} \int 4 + 2(\cos\theta_1 + \cos\theta_2 + 1)dM & \text{d even,} \\ \int -4\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}dM & \text{d odd.} \end{array} \tag{7.14}$$

For $d < 4$ the expression above and below should be modified to drop terms that cannot contribute to the pole in x.

Define

$$I_0[j_1, j_2] = 4\Delta_{j_1}^0\Delta_{j_2}^0 + \Delta_0^{j_1-1}\Delta_0^{j_2} + \Delta_0^{j_2-1}\Delta_0^{j_1}. \tag{7.15}$$

With $I = I_1 + I_0$, the contribution from the second term is

$$I_2[d, j_1, j_2] = I[d, j_1, j_2] - I[d, j_1, j_2+1] - I[d, j_1+1, j_2] + I[d, j_1+1, j_2+1]. \tag{7.16}$$

Finally we consider the remaining quadratic term in (7.3). We will call $\sum_i q_i^k = 4\cos\frac{k\theta_1}{2}\cos\frac{k\theta_2}{2} = \alpha_k$, to save space.

$$-\log(1-(\alpha_k x^k - x^{2k})) = \sum_n \frac{(\alpha_k x^k - x^{2k})^n}{n} = \sum_{p,q}(-1)^q \alpha_k^p x^{(p+2q)k}\frac{1}{p+q}\binom{p+q}{q}. \tag{7.17}$$

Again we will want to collect the coefficient of x^d here. A term in the sum above contributes to this coefficient only if $p + 2q = d/k$. Also, this expression is summed over k against $\phi(k)/k$. This means we need to consider the sum

$$\sum_{k|d}\sum_{p=0}^{d/k} \frac{\phi(k)}{k}\alpha_k^p(-1)^{(\frac{d}{k}-p)/2}\frac{2}{\frac{d}{k}+p}\binom{\frac{d}{2k}+p/2}{p}. \tag{7.18}$$

We now look at a generic integral, integrating this term against $\cos\frac{A\theta_1}{2}\cos\frac{B\theta_2}{2}$. All terms occurring in the actual integral of the term in (7.17) may be reduced to this form. Use the identity

$$\int 4^p\left[\cos\frac{k\theta_1}{2}\right]^p\cos\frac{A\theta_1}{2}\left[\cos\frac{k\theta_2}{2}\right]^p\cos\frac{B\theta_2}{2} = \binom{p}{\frac{1}{2}(p-\frac{A}{k})}\binom{p}{\frac{1}{2}(p-\frac{B}{k})}. \tag{7.19}$$

To shorten expressions, define $\frac{p}{2} = s, \frac{d}{2k} = l, \frac{A}{2k} = m, \frac{B}{2k} = n$.

This allows us to write the generic integral over the sum in (7.18) as

$$S_1[d, A, B] = \sum_{k|(d,A,B)} \phi(k) \sum_{s=max(m,n)}^{l} (-1)^{l-s} \binom{s+l}{2s} \binom{2s}{s-m} \binom{2s}{s-n} \frac{1}{k(l+s)}.$$
(7.20)

Now define

$$\begin{aligned}
S_2[d, A, B] &= S_1[d, A, B] + S_1[d-4, A, B] \\
&\quad - \sum_{\sigma_1, \sigma_2 = \pm 1/2} \left(S_1[d-1, A+\sigma_1, B+\sigma_2] + S_1[d-3, A+\sigma_1, B+\sigma_2] \right) \\
&\quad + \sum_{\rho_1, \rho_2 = \pm 1} S_1[d-2, A+\rho_1, B] + S_1[d-2, A, B+\rho_2] \\
&\quad + 2S_1[d-2, A, B].
\end{aligned}$$
(7.21)

We can now collect all the terms that appear in the integral over the quadratic term (7.17)

$$I_3[d, j_1, j_2] = S_2[d, j_1, j_2] + S_2[d, j_1+1, j_2+1] - S_2[d, j_1+1, j_2] - S_2[d, j_1, j_2+1]. \quad (7.22)$$

Collecting the terms contributing, one finds the following enlightening result:

$$\int dM Z[\theta, \theta_1, \theta_2] \bar{\chi}^*_{[d,j_1,j_2]} = I_2[d, j_1, j_2] + I_3[d, j_1, j_2]. \quad (7.23)$$

where I_2 is defined in (7.16) and I_3 in (6.4). As we noted above, for $d < 4$ the expressions get modified.

These sums are prohibitively difficult to evaluate by hand, but may be easily done with a computer. A *Mathematica* script which evaluates the sum above, with all special cases included, is available online at `http://people.fas.harvard.edu/~llhuang/conformalresult.nb`.

It is easy to list the operators in the theory at low scaling dimension:

Low Dimension Operators in the Pure YM Theory		
Dimension	Operators	Conformal Representation Content
1	no operators	no representations
2	$F_{\mu\nu}$	[2,1,0]+[2,0,1]
3	no primary operators	no representations
4	$F_{\mu\nu} * F^{\mu\nu}$, $|F|^2$	[4,0,0]+[4,0,0]
	$F_{\{\mu\nu} F^{\nu}_{\sigma\}} - |F|^2$	[4,1,1]
	$F_{\{\mu\nu} F_{\rho\sigma\}} - F_{\{\mu\nu} F^{\nu}_{\sigma\}} - |F|^2$	[4,2,0]+[4,0,2]

Now consider large values of d. Neglecting the angular variables, we see that, for large values of d

$$Z = -\sum \frac{\phi(k)}{k} \ln \frac{(1+x)(x^2 - 4x + 1)}{(1-x)^3} \approx \beta^d x^d. \quad (7.24)$$

where $\beta = 2 + \sqrt{3}$ is the larger root of the quadratic term. This is the characteristic Hagedorn growth in the number of states. It is easy to verify, numerically, that (7.23) does reproduce the right growth in the density of states.

As a final consistency check, it is necessary for (7.23) to sum to an integer for every value of d, j_1, j_2; this is not at all apparent from the expression we have. Nevertheless, using some elementary number-theoretic results, we show, in the appendix, that our answer always sums to an integer.

As we reach operators of dimension N^2, trace identites relate multi-trace states to single-trace states. Consequently, the definition of *single-trace* becomes ambiguous. The representations we find can be identified as particles in a dual string theory only at large N.

We list here the representation content of the theory up to dimension 9.

Representation content of pure YM	
Dimension	Representations
2	[2,0,1] + [2,1,0]
3	nothing
4	2[4,0,0] + [4,0,2] + [4,2,0] + [4,1,1]
5	[5,3/2,3/2]
6	2[6,0,0] + 2[6,0,1] + [6,0,3] + 2 [6,1,0] + 2[6,1,1]
	+ [6,1,2] + [6,1,3] + [6,2,1] + [6,2,2] + [6,3,0] + [6,3,1]
7	4[7,1/2,3/2] + 2[7,1/2,5/2] + 4[7,3/2,1/2] + 4[7,3/2,3/2]
	+ 2[7,3/2,5/2] + 2[7,5/2,1/2] + 2 [7,5/2,3/2] + [7,5/2,5/2]
8	6[8,0,0] + 4[8,0,1] + 5[8,0,2] + [8,0,3] + 2 [8,0,4]
	+ 4 [8,1,0]+ 10 [8,1,1] + 7[8,1,2] + 5[8,1,3] + [8,1,4]
	+ 5[8,2,0] + 7 [8,2,1]+ 8 [8,2,2] + 3[8,2,3] + [8,2,4]
	+ [8,3,0]+ 5[8,3,1] + 3[8,3,2] + [8,3,3]+ 2[8,4,0]
	+ [8,4,1] + [8,4,2]
9	14[9,1/2,1/2] + 20[9,1/2,3/2]+ 15[9,1/2,5/2] + 6[9,1/2,7/2]
	+ 20[9,3/2,1/2] + 28[9,3/2,3/2] + 18[9,3/2,5/2] + 7[9,3/2,7/2]
	+ 2 [9,3/2,9/2] + 15[9,5/2,1/2] + 18 [9,5/2,3/2] + 12[9,5/2,5/2]
	+ 4 [9,5/2,7/2] + 6[9,7/2,1/2] + 7 [9,7/2,3/2] + 4 [9,7/2,5/2]
	+ [9,7/2,7/2]+ 2[9,9/2,3/2]

8. $\mathcal{N} = 4$ SYM

In principle, it is not difficult to generalize the procedure above to the case of the $\mathcal{N} = 4$ Yang Mills Theory. This theory has an exact superconformal symmetry. Representations of the Superconformal group are labeled by the highest weight under SO(4) × SO(2) and the *R charges*. These were originally classified in [12]. They were studied in [13 – 15] and are discussed in detail in [16].

It is easy to generalize the partition function by adding in chemical potentials for the R charges. This result can be read off from the appendix in [16]. Similarly, it is simple to generalize the result for the Haar measure and the characters [17].

Unfortunately, short representations in the superconformal case have a rather more intricate structure than in the conformal case and it is not always possible to write them as a difference of two long representations. This complication is not important in the spectrum of single trace operators in the $\mathcal{N} = 4$ theory, because it is known that the only relevant short representations are the $\frac{1}{2}, \frac{1}{2}$ BPS multiplets.

The more serious complication is numerical. Performing the character decomposition involves finding the coefficient of a specified monomial in a given power series expansion. Since we have six chemical potentials in the supersymmetric case, the simple algorithms are $O(d^6)$. Thus, the calculation quickly becomes unfeasible.

In a set of papers [18 – 22], Bianchi et. al. conjectured that the spectrum of the free SYM theory may be obtained from the spectrum of type IIB theory on flat space through a specified algorithm. They verified their conjecture using a sieve procedure which allowed them to determine the spectrum up to scaling dimension 10.

Further verification of this conjecture must await either a deeper understanding of their result or the development of more efficient numerical techniques.

9. Conclusions

Unitary representations of the Conformal Algebra must obey $d \geq j_1 + j_2 + 2$, for $j_1 j_2 \neq 0$ and $d \geq j_1 + j_2 + 1$ otherwise. Depending on whether either of these bounds is saturated, the characters of the conformal group fall into three classes. These are described in (4.3).

The Free Yang Mills theory on a sphere has an exact conformal symmetry. Hence, its partition function may be written as a sum over the characters above. Formally, we have the result

$$Z = \sum N_{d,j_1,j_2} \chi_{d,j_1,j_2}. \tag{9.1}$$

In this note, we performed this decomposition. We find that N_{d,j_1,j_2} is described by (7.23). Our formula demonstrates the correct asymptotics. Moreover, it is possible to prove that it always produces an integer.

Acknowledgments

A. Proof of integral multiplicities

Consider (7.20), which has a sum over k and s. Schematically, the set of allowed k, s, l values is shown below. In the figure, for each value of k, s can range over the values demarcated by the horizontal line at the bottom and the outermost line.
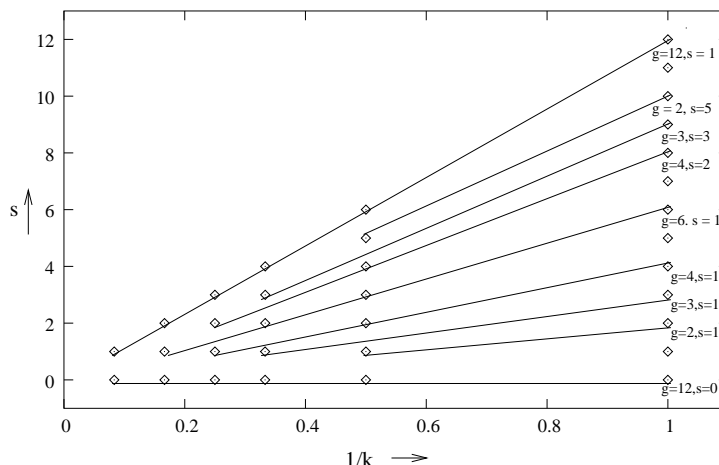
Figure 1: Grouping the terms.

The critical point is to partition this large set of values correctly. We group the set of k, s values in subsets of the kind $S_{l_0,s_0} = \{s, k, l, m, n : l + s = p(l_0 + s_0)\}$ This foliation is indicated on the diagram.

It is clear that in each partition, we have $pk = g$, where g is a constant. Second, we have $\gcd(s_0 + l_0, l_0 - s_0, s_0 - m_0, s_0 - n_0, s_0) = 1$. Since (7.20) can be written as:

$$\frac{\alpha_1}{g(l_0 + s_0)} = \frac{\alpha_2}{2gs_0} = \frac{\alpha_3}{g(l_0 - s_0)} = \frac{\alpha_4}{g(s_0 - m_0)} \cdots$$

where the α_i are integral, it suffices to show that the sum is divisible by g to show that it is an integer.

This leaves us to prove the following statement:

$$\sum_{k|g} \phi(g/k)(-1)^{k(l_0-s_0)} \binom{(s_0 + l_0)k}{2ks_0} \binom{2ks_0}{(s_0 - m_0)k} \binom{2ks_0}{k(s_0 - n_0)} \bmod g = 0. \quad (A.1)$$

For notational simplicity, we consider the simpler statement,

$$\sum_{k|g}(-1)^{n_3 k} \binom{n_1 k}{n_2 k} \phi(g/k) \bmod g = 0, \quad (A.2)$$

where n_i are arbitrary integers with $\gcd n_i = 1$. Furthermore, we take $g = p^t$, where p is prime. We take $p \neq 2$ so that $(-1)^{n_3 k}$ has the same sign. The generalization of this proof to generic g is straightforward.

With $t = 1$, our sum is:

$$\binom{n_1}{n_2}(t - 1) + \binom{n_1 t}{n_2 t} = 0(\bmod t). \quad (A.3)$$

where we have used $\binom{n_1 t}{n_2 t} = \binom{n_1 t}{n_2 t} \bmod t$. Assume the statement is true for $t = n$. For $t = n + 1$, our sum is:

$$\sum_{i=0}^{n+1} \binom{n_1 p^i}{n_2 p^i} \phi(p^{n+1-i}) = p \sum_{i=0}^{n} \binom{n_1 p^i}{n_2 p^i} + \binom{n_1 p^{n+1}}{n_2 p^{n+1}} - \binom{n_1 p^n}{n_2 p^n}. \quad (A.4)$$

– 14 –

The first term is divisible by p^{n+1} by hypothesis. With $n_3 = n_1 - n_2$, write the second term as:

$$\binom{n_1 p^n}{n_2 p^n} \left(\frac{n_1 p^{n+1}(n_1 p^{n+1} - 1) \cdots (n_1 p^n + 1) - n_2 p^{n+1} \cdots (n_2 p^n + 1) n_3 p^{n+1} \cdots (n_3 p^n + 1)}{n_2 p^{n+1} \cdots n_2 p^n n_3 p^{n+1} \cdots n_3 p^n} \right).$$

We can cancel leading terms divisible by p^{n+1} in the numerator and denominator, but then we notice that subleading terms not divisible by p^{n+1} cancel in the numerator but not in the denominator. So the second term is also divisible by p^{n+1}. This proves our result.

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [`hep-th/9711200`].

[2] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, Phys. Rept. **323** (2000) 183 [`hep-th/9905111`].

[3] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Van Raamsdonk, *The hagedorn/deconfinement phase transition in weakly coupled large-N gauge theories*, Adv. Theor. Math. Phys. **8** (2004) 603 [`hep-th/0310285`].

[4] B. Sundborg, *The hagedorn transition, deconfinement and $N = 4$ sym theory*, Nucl. Phys. **B 573** (2000) 349 [`hep-th/9908001`].

[5] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang-Mills*, JHEP **04** (2002) 013 [`hep-th/0202021`].

[6] N. Beisert, *The dilatation operator of $N = 4$ super Yang-Mills theory and integrability*, Phys. Rept. **405** (2005) 1 [`hep-th/0407277`].

[7] I. Bars and M. Günaydin, *Unitary representations of noncompact supergroups*, Commun. Math. Phys. **91** (1983) 31.

[8] N. Beisert, G. Ferretti, R. Heise and K. Zarembo, *One-loop QCD spin chain and its spectrum*, Nucl. Phys. **B 717** (2005) 137 [`hep-th/0412029`].

[9] S. Minwalla, *Restrictions imposed by superconformal invariance on quantum field theories*, Adv. Theor. Math. Phys. **2** (1998) 781 [`hep-th/9712074`].

[10] G. Mack, *All unitary ray representations of the conformal group* $SU(2, 2)$ *with positive energy*, Commun. Math. Phys. **55** (1977) 1.

[11] W. Fulton and J. Harris, *Representation theory: a first course*, Springer-Verlag, Berlin 1991.

[12] V.K. Dobrev and V.B. Petkova, *All positive energy unitary irreducible representations of extended conformal supersymmetry*, Phys. Lett. **B 162** (1985) 127.

[13] V.K. Dobrev and V.B. Petkova, *On the group theoretical approach to extended conformal supersymmetry: classification of multiplets*, Lett. Math. Phys. **9** (1985) 287.

[14] V.K. Dobrev and V.B. Petkova, *Group theoretical approach to extended conformal supersymmetry: function space realizations and invariant differential operators*, Fortschr. Phys. **35** (1987) 537.

[15] V.K. Dobrev and V.B. Petkova, proceedings, A.O. Barut and H.D. Doebner eds., Lecture Notes in Physics, **261**, Springer-Verlag, Berlin 1986, pp. 300–308.

[16] F.A. Dolan and H. Osborn, *On short and semi-short representations for four dimensional superconformal symmetry*, *Ann. Phys. (NY)* **307** (2003) 41 [`hep-th/0209056`].

[17] V.K. Dobrev, *Characters of the positive energy uirs of $D = 4$ conformal supersymmetry*, `hep-th/0406154`.

[18] M. Bianchi, J.F. Morales and H. Samtleben, *On stringy $AdS_5 \times S^5$ and higher spin holography*, *JHEP* **07** (2003) 062 [`hep-th/0305052`].

[19] N. Beisert, M. Bianchi, J.F. Morales and H. Samtleben, *On the spectrum of AdS/CFT beyond supergravity*, *JHEP* **02** (2004) 001 [`hep-th/0310292`].

[20] N. Beisert, M. Bianchi, J.F. Morales and H. Samtleben, *Higher spin symmetry and $N = 4$ SYM*, *JHEP* **07** (2004) 058 [`hep-th/0405057`].

[21] M. Bianchi, *Higher spin symmetry (breaking) in $N = 4$ SYM and holography*, *Comptes Rendus Physique* **5** (2004) 1091 [`hep-th/0409292`].

[22] M. Bianchi, *Higher spins and stringy $AdS_5 \times S^5$*, *Fortschr. Phys.* **53** (2005) 665 [`hep-th/0409304`].